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Domination Cover Pebbling Number for Sun

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Abstract

Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The domination cover pebbling number, ψ(G), of a graph G is the minimum number of pebbles that are placed on V(G) such that after a sequence of pebbling moves, the set of vertices with pebbles forms a dominating set of G, regardless of the initial configuration. In this paper, we determine ψ(G) for Sun.

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1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of graph pebbling [5]. There have been many developments since Hulbert's survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from V(G) to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected. A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, π(G), to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v, a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration.

The domination cover pebbling [3] is the combination of two ideas cover pebbling [2] and domination [4]. This introduces a new graph invariant called the domination cover pebbling number, ψ(G). Recall that, a set of vertices D in G is a dominating set if every vertex in G is either in D or adjacent to a vertex of D. The cover pebbling number , λ(G), is defined as the minimum number of pebbles required such that given any initial configuration of at least λ(G) pebbles, it is possible to make a series of pebbling moves to place at least one pebble on every vertex of G. The domination cover pebbling number of a graph G, proposed by A. Teguia, is the minimum number ψ(G) of pebbles required so that any initial configuration of at least ψ(G) pebbles can be transformed so that the set of vertices that contain pebbles form a dominating set of G. In [3], Gardner et al. computed domination cover pebbling number for complete r-partite graph, path, wheel graph, cycle, and binary tree. We have also determined the domination cover pebbling number of some families of graph in [7-10]. We now proceed to determine the domination cover pebbling number for sun.

2. Domination cover pebbling number for sun

Definition 2.1 [6] A simple connected graph G is called a SUN if it contains exactly one cycle C_n and R distinct vertices not in C_n such that each $v_p \in R$ is connected to exactly one $v_i \in C_n$. A sun is denoted $C_n.\ R_I^{(d)}$, where $v_i \in I$ iff $\{v_iv_p\} \in E(G)$ and (d) = max{(d(v_{ia}, v_{ib}))} for all $v_{ia}, v_{ib} \in I$.

Now we give the labeling for $C_n.\ R_I^{(d)}$. For $C_n : v_0v_1v_2 \ldots v_{k-1}v_ku_{k-1} \ldots u_1v_0$ if n = 2k (k ≥ 2) and $C_n : v_0v_1v_2 \ldots v_{k-1}u_{k-1} \ldots u_1v_0$ if n = 2k – 1 (k ≥ 2) and then label the vertices of R by, v_i' if $\{v_i v_i'\} \in E(G)$ $i \neq 0$ and v_t if $\{v_tv_0\} \in E(G)$ and u_j' if $\{u_j u_j'\} \in E(G)$, where $G = C_n.\ R_I^{(d)}$.

Lemma 2.2 The value of $\psi(C_n. R_I^{(d)})$ is attained when the original configuration consists of placing all the pebbles on any one of the vertices of R.

Proof: Assume first that a worst configuration consists of more than one set of consecutively pebbled vertices (islands) in C_n and no pebbled vertices in R. The cardinality of any such island in C_n is at most two, for were it to be three or more, one could relocate the pebbles to the inner one or two vertices, and there by causing a larger number of pebbles to be needed to cover dominate the vertices of $C_n. R_I^{(d)}$ - a contradiction. Thus each "island" consists of at most two vertices. Now consider the effect of relocating all the pebbles onto a single island. Once again one reaches a contradiction to the assumption that there could be more than one island since after relocating all the pebbles to a single island, one would now require more pebbles to cover dominate the vertices of the graph. Next, assume that island in C_n consists of exactly two vertices. Once again, we consider the effect of relocating all the pebbles to any one of the two pebbled vertices of C_n. One would now require more pebbles to cover dominate the vertices of the graph. So, our assumption is wrong.

Next, assume that a worst configuration consists of exactly one pebbled vertex in C_n and more than one pebbled vertex in R. Now consider the effect of relocating all the pebbles in the vertex of C_n to a vertex of R. One would now require more pebbles to cover dominate the vertices of the graph – a contradiction. From this, we conclude that any worst configuration never consists of a pebbled vertex in C_n.

Finally, assume that a worst configuration consists of more than one pebbled vertex in R. Now, relocate all the pebbles from the vertices of R onto a single vertex of R. Once again one reaches a contradiction to the assumption that there could be more than one pebbled vertex, since after relocating all the pebbles to a single vertex of R , one would now require more pebbles to cover dominate the vertices of the graph $C_n. R_I^{(d)}$. The statement follows. ■

Since placing all the pebbles on a single vertex of R is a worst configuration, we will now determine the value of $\psi(C_n. R_I^{(d)})$

Theorem 2.3 Let $C_n.\,R_I^{(d)}$ be the sun. Then

$$\psi(C_n.\,R_I^{(d)}) = \max\left\{ \sum_{v_i \in I} 2^{i+1} + \sum_{u_j \in I} 2^{j+1} + \sum_{v_k \notin I} 2^{k+1}\mu(P_v) + \sum_{u_l \notin I} 2^{l+1}\mu(P_u) - \gamma(v_t) \right\} \text{ for all}$$

$v_t \in R$, where v_k and u_l are the initial vertices of the paths P_v and P_u respectively and $\gamma(v_t) = 1$ if both v_1 and u_1 are pebbled and $\gamma(v_t) = 0$ if u_1 or v_1 or both are unpebbled. Also,

$$\mu(P_v) = \begin{cases} 0 & \text{if } v = 0 \text{ or } 1 \text{ or } 2 \\ \psi(P_v) & \text{if } v \geq 3 \end{cases}$$

and $$\mu(P_u) = \begin{cases} 0 & \text{if } u = 0 \text{ or } 1 \text{ or } 2 \\ \psi(P_u) & \text{if } u \geq 3 \end{cases}.$$

Proof: We first prove this theorem for n=2m.

Case 1: Consider the Sun $C_n.\,R_I^{(d)}$ where n = 2m ($m \geq 2$). By Lemma 2.2, fix a target vertex from R. Label the target vertex by v_t and then label the remaining vertices as given in Definition 2.1. Now, we have to put one pebble each at the vertices v_i and u_j which belong to I, to cover dominate the vertices v_i' and u_j' respectively. For that, we need 2^{i+1} and 2^{j+1} pebbles at v_t (since, $d(v_t, v_i) = i + 1$; $d(v_t, u_j) = j + 1$) for the vertices v_i and u_j respectively. So, totally we need $\sum_{v_i \in I} 2^{i+1} + \sum_{u_j \in I} 2^{j+1}$ pebbles to cover dominate the vertices which are not in C_n for all $v_i, u_j \in I$. Note that, in this process, we have also cover dominated the vertices v_{i-1}, and v_{i+1} by putting a pebble at v_i and we have cover dominated the vertices u_{j-1}, and u_{j+1} by putting a pebble at u_j. Next, we have to cover dominate the remaining vertices which are in between the neighbor vertices of I.

For that, consider the paths P_A: $v_0v_1 \ldots v_{m-1}v_m$ and P_B: $v_0u_1u_2 \ldots u_{m-1}$. First, consider the path P_A: $v_0v_1 \ldots v_{m-1}v_m$ and two consecutive vertices which belong to I, say v_i and v_{i+h} ($i \geq 0$ and $h > i$) in P_A. If both v_i and v_{i+h} are adjacent or connected by at most two vertices in P_A then we have already cover dominated the vertices between v_i and v_{i+h}. Otherwise consider the path P_V : $v_{i+1}v_{i+2} \ldots v_{i+h-2}$. So, we need 2^{i+2} (ψ (P_V)) pebbles at v_t to cover dominate the vertices of the path P_v , since $d(v_t, v_{i+1})=i+2$. If $v_m \in$ I then the path P_v ends with the vertex v_{m-2}, otherwise, the path P_v ends with v_{m-1} in P_A, if $u_{m-1} \notin$ I (this case arises for the last two consecutive vertices which belong to I in P_A). After this process, we would have cover dominated all the vertices of the path P_A , using $\sum_{v_i \in I} 2^{i+1} + \sum_{v_k \notin I} 2^{k+1} \mu(P_v)$ pebbles from v_t, where v_k is the initial vertex of the path P_v and

$$\mu(P_v) = \begin{cases} 0 & , if\ v=0\ or\ 1\ or\ 2 \\ \psi(P_v) & , if\ v \geq 3 \end{cases}.$$

Now, we do the same thing for the other path $P_B : v_0u_1u_2 \ldots u_{m-1}v_m$. To cover dominate the vertices of the path P_B, we need $\sum_{u_j \in I} 2^{j+1} + \sum_{u_l \notin I} 2^{l+1} \mu(P_u)$ pebbles at v_t where u_l is the initial vertex of the path P_u and

$$\mu(P_u) = \begin{cases} 0 & , if\ u=0\ or\ 1\ or 2 \\ \psi(P_u) & , if\ u \geq 3 \end{cases}$$

Now, we have cover dominated all the vertices of $C_n.\ R_I^{(d)}$, but we may use an extra pebble at v_0 if both v_1 and u_1 are pebbled by the previous processes. Thus we have to subtract one pebble from v_t. Otherwise, we are already done.

That is, $\gamma(v_t) = \begin{cases} 1 & if\ both\ v_1\ and\ u_1\ are\ pebbled \\ 0 & if\ either\ v_1\ or\ u_1\ or\ both\ are\ unpebbled \end{cases}$

Thus we need, $\sum_{v_i \in I} 2^{i+1} + \sum_{u_j \in I} 2^{j+1} + \sum_{v_k \notin I} 2^{k+1} \mu(P_v) + \sum_{u_l \notin I} 2^{l+1} \mu(P_u) - \gamma(v_t)$

pebbles at v_t to cover dominate the vertices of $C_n.\ R_I^{(d)}$

Now, consider another target vertex in R and label it by v_t and then carry out the process described above. Finally, we choose the one from R, which takes maximum number of pebbles to cover dominate the vertices of $C_n.\ R_I^{(d)}$.

That is,

$$\Psi(C_n.R_I^{(d)}) = \max\left\{ \sum_{v_i\in I} 2^{i+1} + \sum_{u_j\in I} 2^{j+1} + \sum_{v_k\notin I} 2^{k+1}\mu(P_v) + \sum_{u_l\notin I} 2^{l+1}\mu(P_u) - \gamma(v_t) \right\}$$

for all $v_t \in R$.

Case 2: Consider the sun $C_n.R_I^{(d)}$, where n = 2m – 1 (m ≥ 2).

By Lemma 2.2, fix a target vertex v_t from the R distinct vertices not in C_n. Label the target vertex by v_t and then label the remaining vertices as given in Definition 2.1. Using $\sum_{v_i\in I} 2^{i+1} + \sum_{u_j\in I} 2^{j+1}$ pebbles at v_t, we can cover dominate all the vertices outside C_n for all v_i, $u_j \in I$.

Also, we have cover dominated the adjacent vertices of v_i and u_j in C_n. Next, we have to cover dominate the remaining vertices in between the neighbor vertices of I.

Case (2a): $v_{m-1} \in I$ and $u_{m-1} \notin I$.

Consider the paths P_A: $v_0v_1v_2 \ldots v_{m-1}$ and P_B : $v_0u_1u_2 \ldots u_{m-1}$. First consider the path P_A and two consecutive vertices which belong to I, say, v_i and v_{i+h} (i ≥ 0 and h > i). So, we need $\sum_{v_k\notin I} 2^{k+1}\mu(P_v)$ pebbles at v_t to cover dominate the vertices of P_A, where P_v:$v_{i+1}, v_{i+2}, \ldots, v_{i+h-2}$ and $v_k = v_{i+1}$ is the initial vertex of the path P_v. Since, $v_{m-1} \in I$, the vertex u_{m-1} is already cover dominated. A similar work can be done for the other path P_B, that is, if $u_{m-2} \in I$, then the path P_u ends with u_{m-2}. Otherwise, P_u ends with u_{m-3}. Therefore, to cover dominate the vertices of the path P_B we need $\sum_{u_l\notin I} 2^{l+1}\mu(P_u)$ pebbles at v_t.

Case (2b): $u_{m-1} \in I$ and $v_{m-1} \notin I$.

Consider the paths P_A : $v_0v_1v_2 \ldots v_{m-1}$ and P_B : $v_0u_1u_2 \ldots u_{m-1}$. First consider the path P_B and two consecutive vertices which belong to I, say, u_i and u_{i+h} (i ≥ 0 and h > I, Here let $u_0 = v_0$). So, we need $\sum_{u_l\notin I} 2^{l+1}\mu(P_u)$ pebbles at v_t to cover dominate the vertices of P_B. A similar work can be done for the other path P_A(as described in case(2a)). Thus, to cover dominate the vertices of the path P_A we need $\sum_{v_k\notin I} 2^{k+1}\mu(P_v)$ pebbles at v_t.

Case (2c): Both v_{m-1} and $u_{m-1} \in I$.

Consider the paths $P_A : v_0v_1 \ldots v_{m-1}$ and $P_B : v_0u_1u_2 \ldots u_{m-1}$. Clearly, the paths P_u and P_v end with v_{m-1} and u_{m-1} respectively. So, we need $\sum_{v_k \notin I} 2^{k+1}\mu(P_v)$ to cover dominate the vertices of the path P_A and $\sum_{u_l \notin I} 2^{l+1}\mu(P_u)$ to cover dominate the vertices of the path P_B at v_t.

Case (2d): Both v_{m-1} and $u_{m-1} \notin I$.

Consider the paths $P_A : v_0v_1 \ldots v_{m-1}$ and $P_B : v_0u_1u_2 \ldots u_{m-1}$. Clearly, the paths P_u and P_v end with v_{m-2} and u_{m-2} respectively. So, we need $\sum_{v_k \notin I} 2^{k+1}\mu(P_v) + \sum_{u_l \notin I} 2^{l+1}\mu(P_u)$ pebbles to cover dominate the vertices of the paths P_A and P_B.

Now, we cover dominate all the vertices of $C_n.R_I^{(d)}$.But there may be an extra pebble at v_0 if both v_1 and u_1 are pebbled by the previous processes. So, we have to subtract one pebble from v_t. Otherwise, we are already done. That is,

$$\gamma(v_t) = \begin{cases} 1 & \text{if both } v_1 \text{ and } u_1 \text{ are pebbled} \\ 0 & \text{if either } v_1 \text{ or } u_1 \text{ or both are unpebbled} \end{cases}$$

Thus, we need $\sum_{v_i \in I} 2^{i+1} + \sum_{u_j \in I} 2^{j+1} + \sum_{v_k \notin I} 2^{k+1}\mu(P_v) + \sum_{u_l \notin I} 2^{l+1}\mu(P_u) - \gamma(v_t)$ pebbles at v_t to cover dominate the vertices of $C_n.R_I^{(d)}$.

Now, consider another target vertex in R and label it by v_t and then carry out the process described above. Finally, we choose the one from R, which takes maximum number of pebbles to cover dominate the vertices of $C_n.R_I^{(d)}$.

Therefore, we get,

$$\psi(C_n.R_I^{(d)}) = \max\left\{\sum_{v_i \in I} 2^{i+1} + \sum_{u_j \in I} 2^{j+1} + \sum_{v_k \notin I} 2^{k+1}\mu(P_v) + \sum_{u_l \notin I} 2^{l+1}\mu(P_u) - \gamma(v_t)\right\}$$

for all $v_t \in R$. ■

Corollary 2.4 For R = n, ψ(C_n. $R_I^{(d)}$) = $\begin{cases} 3.2^{m+1} - 7, \text{ if } n = 2m (m \geq 2) \\ 2^{m+2} - 7, \text{ if } n = 2m-1 (m \geq 2) \end{cases}$,

For R = n - 1, ψ(C_n. $R_I^{(d)}$) = $\begin{cases} 3.2^{m+1} - 10, \text{ if } n = 2m (m \geq 2) \\ 2^{m+2} - 10, \text{ if } n = 2m-1 (m \geq 2) \end{cases}$.

Proof:

Case A: Let R = n. Without loss of generality, choose a vertex $v_t \in$ R.

Case A1: Let n = 2m (m ≥ 2). This implies that d(v_t, v_m) = m + 1. Note, μ(P_v) = μ(P_u) = 0, since v = 0 = u.

So, ψ(C_n. $R_I^{(d)}$) = $\sum_{\substack{i=0 \\ v_i \in I}}^{m} 2^{i+1} + \sum_{\substack{j=1 \\ u_j \in I}}^{m-1} 2^{j+1} - \gamma(v_t)$

= 2+2^{m+1}+ $2\left(\sum_{\substack{i=1 \\ v_i \in I}}^{m-1} 2^{i+1}\right)$ - 1

= 3. 2^{m+1} – 7,

where the second equality follows since both v_1 and $u_1 \in$ I.

Therefore, ψ(C_n. $R_I^{(d)}$) = 3. 2^{m+1} – 7

Case A2: Let n = 2m – 1 (m ≥ 2). This implies that d(v_t, v_{m-1}) = d(v_t, u_{m-1}) = m. Also, μ(P_v) = μ(P_u) = 0, since v = u = 0.

So, ψ(C_n. $R_I^{(d)}$)= $\sum_{\substack{i=0 \\ v_i \in I}}^{m-1} 2^{i+1} + \sum_{\substack{j=1 \\ u_j \in I}}^{m-1} 2^{j+1} - \gamma(v_t)$

= 2+ $2\sum_{i=1}^{m-1} 2^{i+1} - 1$

= 2.2^{m+1} – 7,

where the second equality follows since both v_1 and $u_1 \in$ I.

Therefore, $\psi(C_n.\mathrm{R}_{\mathrm{I}}^{(\mathrm{d})}) = 2^{m+2} - 7$.

Case B: Let R = n – 1. This implies that, one vertex does not belong to I. First label the vertex by v_1 and then label the remaining vertices of $C_n.\mathrm{R}_{\mathrm{I}}^{(\mathrm{d})}$, so that we get maximum number of pebbles at v_t. Also, note $\mu(P_v) = 0$ (since, v = 0 or 1) and $\mu(P_u) = 0$ (since, u = 0 or 1).

Case B1: Let n = 2m (m ≥ 2).

$$\text{So, } \psi(C_n.\mathrm{R}_{\mathrm{I}}^{(\mathrm{d})}) = 2 + \sum_{i=2}^{m} 2^{i+1} + \sum_{j=1}^{m-1} 2^{j+1} - \gamma(v_t)$$

$$= 2 + 2^{m+1} + \sum_{i=1}^{m-1} 2^{i+1} + \sum_{j=1}^{m-1} 2^{j+1} - \gamma(v_t) - 4$$

$$= 2^{m+1} + 2\sum_{i=1}^{m-1} 2^{i+1} + 2 - 4$$

$$= 3(2^{m+1}) - 10,$$

where the third equality follows since $v_1 \notin I$.

Therefore, $\psi(C_n.\mathrm{R}_{\mathrm{I}}^{(\mathrm{d})}) = 3(2^{m+1}) - 10$.

Case B2: Let n = 2m – 1.

$$\text{So, } \psi(C_n.\mathrm{R}_{\mathrm{I}}^{(\mathrm{d})}) = 2 + \sum_{i=2}^{m-1} 2^{i+1} + \sum_{j=1}^{m-1} 2^{j+1}$$

$$= 2^{m+2} - 10,$$

where the first equality follows since $v_1 \notin I$.

Therefore, $\psi(C_n.\mathrm{R}_{\mathrm{I}}^{(\mathrm{d})}) = 2^{m+2} - 10$. ■

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